Problem Set 3: Differential Calculus Solutions
CS&SS Math Camp 2021

1. Plot the function $f(x) = 3x + 2$.

   (a) By eye, what is the derivate of this function at x=4?
   The slope of the line is 3 since it has the form $y = mx + b$ and we know m is the slope.

   (b) Compute the derivative using the appropriate formula.
   $f^{'}(x) = 3 \cdot 1x^{1-1} + 0 = 3$ [By the power rule.]

   Compute the derivative:

2. $f(x) = x^5$
   $f^{'}(x) = 5x^4$ [Use the power rule.]

3. $f(x) = 10x - 30$
   $f^{'}(x) = 10$ [Use the power rule or recall the derivative of any line is m.]

4. $f(x) = 2x^4 + x^2$
   $f^{'}(x) = 2(4x^3) + 2x = 8x^3 + 2x$ [Use the power rule and sum rule.]

5. $f(x) = tan(x) = \frac{sin(x)}{cos(x)}$

$$
\begin{aligned}
f^{'}(x) =& \frac{cos(x)cos(x) - (-sin(x)sin(x))}{cos^2(x)} && \text{[Use quotient rule.]} \\
=& \frac{cos^2(x) + sin^2(x)}{cos^2(x)} \\
=& \frac{1}{cos^2(x)}
\end{aligned}
$$

6. $f(x) = e^{\sin(x)}$

$$g(x) = e^x \quad g'(x) = e^x \quad h(x) = \sin(x) \quad h'(x) = \cos(x)$$

$$\begin{aligned} f'(x) =& g'(h(x)) \cdot h'(x) && \text{[Use the chain rule.]} \\ =& e^{\sin(x)} \cdot \cos(x) \end{aligned}$$

7. $f(x) = xe^x + \log(\sin(x))$

   We need to apply the product rule to the first term, $g(x)$, and the chain rule to the second term, $h(x)$, then the sum rule to the whole term, $f(x) = g(x) + h(x)$.

$$g(x) = xe^x = k(x) \cdot l(x) \Rightarrow k(x) = x \quad k'(x) = 1 \quad l(x) = e^x \quad l'(x) = e^x$$

$$\mathbf{g}'(\mathbf{x}) = l'(x) \cdot k(x) + l(x) \cdot k'(x) = e^x \cdot x + e^x \cot 1 = \mathbf{e^x(x+1)}$$

$$h(x) = \log(\sin(x)) = s(t(x)) \Rightarrow s(x) = \log x \quad s'(x) = \frac{1}{x} \quad t(x) = \sin(x) \quad t'(x) = \cos(x)$$

$$\mathbf{h}'(\mathbf{x}) = s'(t(x)) \cdot t'(x) = \frac{1}{\sin(x)} \cdot \cos(x) = \frac{\cos(\mathbf{x})}{\sin(\mathbf{x})} = \cot(\mathbf{x})$$

$$\mathbf{f}'(\mathbf{x}) = g'(x) + h'(x) = \mathbf{e^x(x+1)} + \frac{\cos(\mathbf{x})}{\sin(\mathbf{x})}$$

   We can also have a function of a different variable. This is just changing the variable name and you will see this a lot in your statistics methods classes.

8. Compute the derivative of $g(\theta) = \theta^2 - \theta^4$

$$g^{'}(\theta) = 2\theta - 4\theta^3 \text{ [Use power and sum rules.]}$$

9. Find the global minimum of $f(z) = z^2 - 6z + 8$

   First compute the derivative, $f'(z)$, and set it equal to zero to find the z value that makes $f'(z) = 0$.

$$\begin{aligned} f^{'}(z) = 2z - 6 = 0 && \text{[Use power and sum rules.]} \\ 2z = 6 && \text{[Add 6 to both sides.]} \\ z = 3 && \text{[Divide both sides by 2.]} \end{aligned}$$

   Check whether it's a min or max by finding the second derivative, $f''(z)$, and seeing whether $f''(z)$ is positive or negative at our critical value,3, i.e. is $f'(3) > 0$ or $f'(3) < 0$?

$$f''(z) = 2 \qquad \text{[Use the power rule.]} f''(3) = 2 > 0$$

The second derivative is constant, equal to 2 at every point $z \in \mathbb{R} = (-\infty, \infty)$. So, at our critical point, $z = 3$, the second derivative is positive and we know that we have a minimum at $z = 3$. Since it's the only critical point, it's the global minimum. The minimum function value is

$$f(3) = 3^2 - 6 \cdot 3 + 8 = -1$$

, i.e. the minimum occurs at the coordinate $(3, -1)$.

10. In the following function, treat x as a constant (i.e. the same way you would treat the number 3 in the following equation $f(x) = 3x^2$, and differentiate with respect to μ:

$$h(\mu) = x\mu^2$$

$$h^{'}(\mu) = x(2 \cdot \mu) = 2x\mu \text{ [Use the power rule.]}$$

11. In the following function of μ, treat $X_1, X_2, ..., X_n$ as constants. Maximize the function over μ. In other words, find the value of μ, expressed in terms of $X_1, X_2, ..., X_n$, at which the function reaches its global maximum:
$L(\mu) = log(\frac{1}{\sqrt{2\pi}} e^{-\sum_{i=1}^{n}(X_i-\mu)^2})$

First, we'll simplify using rules of the logarithms:

$$\begin{aligned} L(\mu) &= log\left(\frac{1}{\sqrt{2\pi}}\right) + log\left(e^{-\sum_{i=1}^{n}(X_i-\mu)^2}\right) \\ &= log\left(\frac{1}{\sqrt{2\pi}}\right) - \sum_{i=1}^{n}(X_i - \mu)^2 \cdot log(e) \\ &= log\left(\frac{1}{\sqrt{2\pi}}\right) - \sum_{i=1}^{n}(X_i - \mu)^2 \end{aligned}$$

The first term is a constant so the derivative is zero. The right-hand term is a sum, and the derivative of the sum is a sum of the derivatives. For each term $(X_i - \mu)^2$, we need to apply the chain rule:

$$L^{'}(\mu) = -\sum_{i=1}^{n}(-2)(X_i - \mu) = 0$$

$$2\sum_{i=1}^{n}(X_i - \mu) = 0$$

$$\sum_{i=1}^{n}(X_i - \mu) = 0$$

$$\sum_{i=1}^{n} X_i - n\mu = 0$$

$$\mu = \frac{1}{n}\sum_{i=1}^{n} X_i$$

To verify it's a max, take the second derivative:

$$L^{'}(\mu) = 2\sum_{i=1}^{n}(X_i - \mu) = 2\sum_{i=1}^{n} X_i - 2n\mu$$

$$L^{''}(\mu) = -2n < 0 \text{ when } n > 0$$

So it's a max, assuming $n > 0$. Since it's the only critical point, it's the global max. This is the maximum likelihood estimate (MLE) of the expectation of a normal distribution, using a sample of size n. The MLE is the sample mean.