Problem Set 4: Integral Calculus Solutions
CS&SS Math Camp 2021

1. (a) Graph the function defined by:

$$f(x) = \begin{cases} \frac{1}{10} & \text{if } x \in [0, 10] \\ 0 & \text{otherwise} \end{cases}$$

This is an example of the uniform probability distribution.

(b) By studying the graph and without using calculus, compute the area under the curve on the interval [2,7].

Area of rectangle = length x width = $(7 - 2) \cdot \frac{1}{10} = \frac{5}{10} = \frac{1}{2}$

(c) Now compute the same area using integral calculus.

$$\int_2^7 \frac{1}{10} dx = \frac{1}{10} x \Big|_2^7 = \frac{1}{10} \cdot 7 - \frac{1}{10} \cdot 2 = \frac{1}{2}$$

Integrate, and check by differentiating:

2. $\int x^7 dx = \frac{1}{8} x^8$ [Use the power rule.]

Check: $\frac{d}{dx} \frac{1}{8} x^8 = \frac{1}{8} \cdot 8x^7 = x^7$

3. $\int x^2 + 6x^5 dx = \frac{x^3}{3} + \frac{6x^6}{6} = \frac{x^3}{3} + x^6$ [Use power rule and sum rule.]

Check: $\frac{d}{dx} \frac{x^3}{3} + x^6 = 3\frac{x^2}{3} + 6 \cdot x^5 = x^2 + 6x^5$

4. $\int \frac{1}{x^2}dx = \int x^{-2}dx = \frac{x^{-1}}{-1} = -\frac{1}{x}$ [Use power rule.]

Check: $\frac{d}{dx} - \frac{1}{x} = (-1)(-1)x^{-2} = \frac{1}{x^2}$

5. $\int \frac{1}{x}dx = \log(x)$

Check: $\frac{d}{dx}log(x) = \frac{1}{x}$

6. $\int (3-x)^{10}dx$

$$u = 3 - x \Rightarrow du = -dx \Rightarrow dx = -1du$$

$$\begin{aligned} \int (3-x)^{10}dx =& \int u^{10}(-1du) && \text{[Use } u\text{-substitution.]} \\ =& -1\frac{u^{11}}{11} && \text{[Use power rule.]} \\ =& -\frac{1}{11}(3-x)^{11} && \text{[Replace } u\text{.]} \end{aligned}$$

Check: $\frac{d}{dx} - \frac{1}{11}(3-x)^{11} = -\frac{1}{11} \cdot 11 \cdot (-1) \cdot (3-x)^{10} = (3-x)^{10}$

7. $\int \sqrt{7x+9}dx$

$$u = 7x + 9 \Rightarrow du = 7dx \rightarrow dx = \frac{1}{7}du$$

$$\begin{aligned} \int (7x+9)^{1/2}dx =& \int u^{1/2}(\frac{1}{7}du) && \text{[Use } u\text{-substitution.]} \\ =& \frac{1}{7} \cdot \frac{u^{3/2}}{3/2} = \frac{2}{21}u^{3/2} && \text{[Use power rule.]} \\ =& \frac{2}{21}(7x+9)^{3/2} && \text{[Replace } u\text{.]} \end{aligned}$$

Check: $\frac{d}{dx}\frac{2}{21}(7x+9)^{3/2} = \frac{2}{21} \cdot \frac{3}{2} \cdot (7x+9)^{1/2} \cdot 7 = \frac{2 \cdot 3 \cdot 7}{21 \cdot 2} \cdot (7x+9)^{1/2} = (7x+9)^{1/2}$

8. $\int e^{5x+2}dx$

$$u = 5x + 2 \Rightarrow du = 5dx \Rightarrow dx = \frac{1}{5}du$$

$$\begin{aligned}
\int e^{5x+2}dx &= \int e^u \cdot \frac{1}{5}du && \text{[Use } u\text{-substitution.]} \\
&= \frac{1}{5}e^u && \text{[Use definition of derivative of } e^x\text{.]} \\
&= \frac{1}{5}e^{5x+2} && \text{[Replace } u\text{.]}
\end{aligned}$$

Check: $\frac{d}{dx}\frac{1}{5}e^{5x+2} = \frac{1}{5}e^{5x+2} \cdot 5 = e^{5x+2}$

9. Compute the area under the curve:

$$\int_{0.5}^{1} x(1-x)^2 dx$$

This is an example of the beta distribution, a probability distribution which we'll see later this week.

$$\begin{aligned}
\int_{0.5}^{1} x(1-x)^2 dx &= \int_{0.5}^{1} x(1-x)(1-x)dx = \int_{0.5}^{1} x(1-2x+x^2)dx \\
&= \int_{0.5}^{1} (x - 2x^2 + x^3)dx = \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 \Big|_{0.5}^{1} \\
&= \left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right] - \left[\frac{1}{2}(0.5)^2 - \frac{2}{3}(0.5)^3 + \frac{1}{4}(0.5)^4\right] \\
&= \left[\frac{2^5 \cdot 3}{2^6 \cdot 3} - \frac{2^7}{2^6 \cdot 3} + \frac{2^4 \cdot 3}{2^6 \cdot 3}\right] - \left[\frac{2^3 \cdot 3}{2^6 \cdot 3} - \frac{2^4}{2^6 \cdot 3} + \frac{3}{2^6 \cdot 3}\right] \\
&= \frac{2^5 \cdot 3 - 2^7 + 2^4 \cdot 3 - 2^3 \cdot 3 + 2^4 - 3}{2^6 \cdot 3} \\
&= \frac{5}{192} = 0.02604167
\end{aligned}$$

10. Compute the area under the curve:

$$\int_2^\infty 4e^{-4x} dx$$

This is an example of the exponential probability distribution, which we'll study later.

$$\int_2^\infty 4e^{-4x} dx = 4 \int_2^\infty e^{-4x} dx = 4 \left(\frac{-1}{4} e^{-4x} \right) \Bigg|_2^\infty = -e^{-4x} \Big|_2^\infty$$

$$= \left(\lim_{x\to\infty} -e^{-4x} \right) - (-e^{-4\cdot 2}) = \left(\lim_{x\to\infty} -\frac{1}{e^{4x}} \right) + e^{-8} = 0 + e^{-8} = \frac{1}{e^8} = 0.0003354626$$