Center for Statistics and the Social Sciences Math Camp 2021

Lecture 3: Differential Calculus

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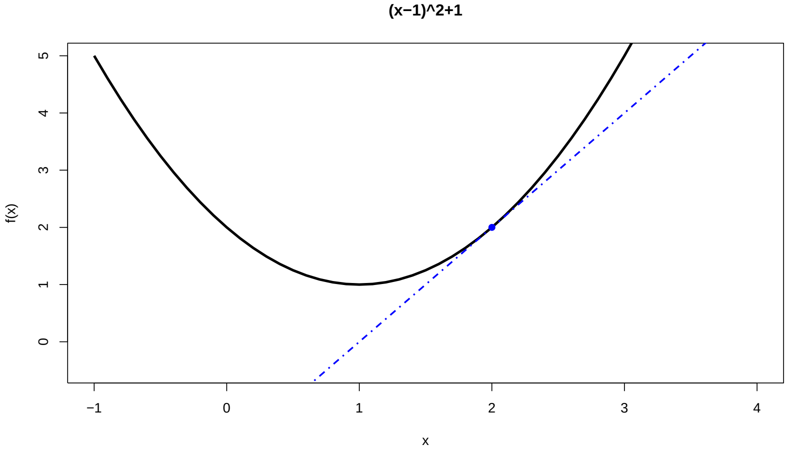
Outline

- Differentiation of functions
  - Defining the derivative
  - Basic differentiation rules
  - Second, third, etc... derivatives
- Critical points of functions
  - What is a critical point?
  - Maximum, minimum, and using the second derivative to tell the difference
- Taylor Series

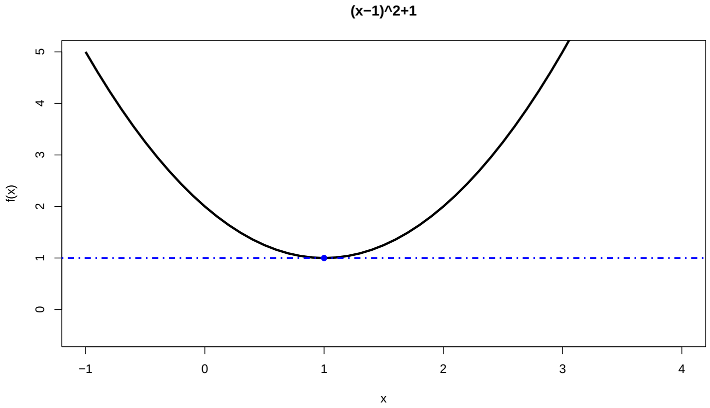
What is the derivative?

The **derivative** of a function can be thought of as the **slope of the line tangent to the curve**, $f(x)$, at the point x. (Skims the curve, touching only at the point x).



Why do we care?

In statistics we are often interested in derivatives to help us find the values that maximize (or minimize) functions. We will be particularly interested in the values x such that the slope of the tangent line is zero.



Finding the Derivative

The derivative of a function $f(x)$ is the **instantaneous rate at which the function is changing** at x. Why would we be interested in finding the derivative of a function?

- growth rate of a population relative to change in time
- change in distance relative to a change in time
- marginal revenue - change in amount of money from item sales relative to change in demand for the items

Think of it as finding the **slope** of a function at specific point. To find the average rate of change over an interval $[a, b]$, we look at the change in $f(x)$ over the length of the interval.

$$\frac{f(b) - f(a)}{b - a}$$

Finding the Derivative

If we want to find the rate of change at a value x, we find the average rate of change over a very small interval (usually of length δ).

$$\frac{f(x+\delta) - f(x)}{\delta}$$

We look at what happens when δ becomes very very small, i.e. when the interval essentially just becomes the point x.

The derivative of $f(x)$ at x is then:

$$lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta}$$

It is denoted by $\frac{d}{dx}f(x)$ or $f'(x)$.

Differentiation with Limits

Given an $f(x)$, how do we find the derivative $f'(x)$?

$$f'(x) = lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta}$$

to start, let's write out the algebra and then take the limit.
Example: $f(x) = mx + b$

$$\begin{aligned}
lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta} &= lim_{\delta \to 0} \frac{m(x+\delta) + b - (mx+b)}{\delta} \\
&= lim_{\delta \to 0} \frac{mx - mx + m\delta + b - b}{\delta} \\
&= lim_{\delta \to 0} \frac{m\delta}{\delta} \\
&= lim_{\delta \to 0} m = m
\end{aligned}$$

Differentiation with Limits

Example: $f(x) = ax^2$

$$
\begin{aligned}
lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta} &= lim_{\delta \to 0} \frac{a(x+\delta)^2 - ax^2}{\delta} \\
&= lim_{\delta \to 0} \frac{a(x^2 + 2x\delta + \delta^2) - ax^2}{\delta} \\
&= lim_{\delta \to 0} \frac{2ax\delta + a\delta^2}{\delta} \\
&= lim_{\delta \to 0} 2ax + a\delta \\
&= 2ax
\end{aligned}
$$

Differentiation Rules

Derivative of a constant:

$$f(x) = a; \;\; f'(x) = 0$$

Derivative of a power:

$$f(x) = ax^n; \;\; f'(x) = n \cdot a \cdot x^{n-1}$$

Derivative of an exponential:

$$f(x) = e^x; \;\; f'(x) = e^x$$

Examples:

- $f(x) = x^4; f'(x) = 4x^3$
- $f(x) = 3x^7; f'(x) = 21x^6$
- $f(x) = 3e^x; f'(x) = 3e^x$

Derivative of an Logarithmic Function:

$$f(x) = log(x); \;\; f'(x) = 1/x$$

Derivative of a Trigonometric Functions:

$$f(x) = sin(x); \;\; f'(x) = cos(x) \quad \& \quad f(x) = cos(x); \;\; f'(x) = -sin(x)$$

Derivative of a Sum of Functions:

$$f(x) = g(x) + h(x); \;\; f'(x) = g'(x) + h'(x)$$

Examples:

- $f(x) = 2log(x); f'(x) = 2/x$
- $f(x) = sin(x) + e^x; f'(x) = cos(x) + e^x$
- $f(x) = 3x^2 + 4x; f'(x) = 6x + 4$

Derivative of the product of two functions:

$$f(x) = g(x) \cdot h(x); \quad f'(x) = g'(x) \cdot h(x) + h'(x) \cdot g(x)$$

Examples:

- $f(x) = x^2 \cdot e^x; f'(x) = 2x \cdot e^x + x^2 \cdot e^x$
- $f(x) = 3x \cdot log(x); f'(x) = 3 \cdot log(x) + 3x \cdot 1/x = 3log(x) + 3$
- $f(x) = 3x \cdot log(x) + 2x; f'(x) = 3log(x) + 3 + 2 = 3log(x) + 5$

Derivative of the division of two functions:

$$f(x) = \frac{g(x)}{h(x)};\ \ f'(x) = \frac{g'(x) \cdot h(x) - h'(x) \cdot g(x)}{h(x)^2}$$

Examples:

- $f(x) = \frac{x^2}{e^x}; f'(x) = \frac{2x \cdot e^x - x^2 \cdot e^x}{e^{2x}} = \frac{2x - x^2}{e^x}$
- $f(x) = \frac{3x}{log(x)}; f'(x) = \frac{3 \cdot log(x) - 3x \cdot 1/x}{log(x)^2} = \frac{3log(x) - 3}{log(x)^2}$

Derivative of a function within a function:

$$f(x) = g(h(x)); \quad f'(x) = g'(h(x)) \cdot h'(x)$$

Examples:

- $f(x) = e^{3x}; g(h) = e^h, h(x) = 3x$
  $g'(h) = e^h, h'(x) = 3 \Rightarrow f'(x) = g'(3x)h'(x) = 3e^{3x}$
- $f(x) = log(1-x); g(h) = log(h), h(x) = 1-x$
  $g'(h) = 1/h, h'(x) = -1 \Rightarrow f'(x) = g'(1-x)h'(x) = \frac{-1}{(1-x)}$
- $f(x) = (2x+2)^2; g(h) = h^2, h(x) = 2x+2$
  $g'(h) = 2h, h'(x) = 2 \Rightarrow f'(x) = g'(2x+2)h'(x) =$
  $2(2x+2) \cdot 2 = 4(2x+2)$

Differentiation rules

Quotient Rule as the Product Rule using the Chain Rule

Derivative of the division of two functions:

$$f(x) = \frac{g(x)}{h(x)} = g(x) \cdot h^{-1}(x)$$

$$f'(x) = g'(x) \cdot h^{-1}(x) + (-1) \cdot h^{-2}(x) \cdot h'(x) \cdot g(x)$$

Example:

- $f(x) = \frac{x^2}{e^x} = x^2 \cdot e^{-x}$;

$$\begin{aligned} f'(x) &= 2x \cdot e^{-x} + (-1) \cdot e^{-2x} \cdot e^x \cdot x^2 \\ &= \frac{2x}{e^x} - \frac{e^x \cdot x^2}{e^{2x}} \\ &= \frac{2x}{e^x} - \frac{x^2}{e^x} \\ &= \frac{2x - x^2}{e^x} \end{aligned}$$

Differentiation rules

Examples

We can combine many rules:

$$f(x) = 3x(2x+1)^4$$

will require the product rule and the chain rule, where $g(x) = 3x$, $k(x) = 2x+1$, and $h(k) = k^4$.

$$\begin{aligned} f'(x) &= g'(x) \cdot h(k) + g(x) \cdot h'(k) \cdot k'(x) \\ &= 3 \cdot (2x+1)^4 + 3x \cdot 4(2x+1)^3 \cdot 2 \\ &= 3(2x+1)^4 + 24x(2x+1)^3 \end{aligned}$$

Second & Third Derivatives

We can find the **second derivative** by taking the derivative of the derivative. The **third derivative** is found by taking the derivative of the second derivative and so on.

The second derivative is the rate of change of the first derivative and can be written as $f''(x)$ or $\frac{d^2}{dx^2}f(x)$.
Example:

$$\begin{aligned} f(x) &= log(4x) \\ f'(x) &= \frac{1}{4x} \cdot 4 = 1/x = x^{-1} \\ f''(x) &= (-1) \cdot x^{-2} = -x^{-2} \\ f'''(x) &= (-2) \cdot -1x^{-3} = 2x^{-3} \end{aligned}$$

Differentiation rules

distance, velocity, acceleration

Let's take d=distance, v=velocity, a=acceleration. You may remember from physics, the distance travel after time t

$$d(t) = \frac{a}{2} t^2$$

The velocity at any time t is the instantaneous rate of change of the distance, $v(t) = d'(t)$:

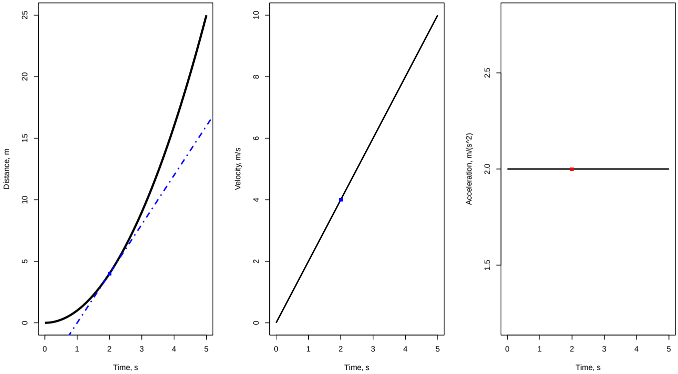
$$v(t) = 2 \cdot \frac{a}{2} t = at$$

The acceleration at any time t is the instantaneous rate of change of the velocity, $a(t) = v'(t) = d''(t)$:

$$a(t) = a$$

Differentiation rules

$$d(t) = t^2 \quad v(t) = d'(t) = 2t \quad a(t) = v'(t) = d''(t) = 2$$



Critical Values

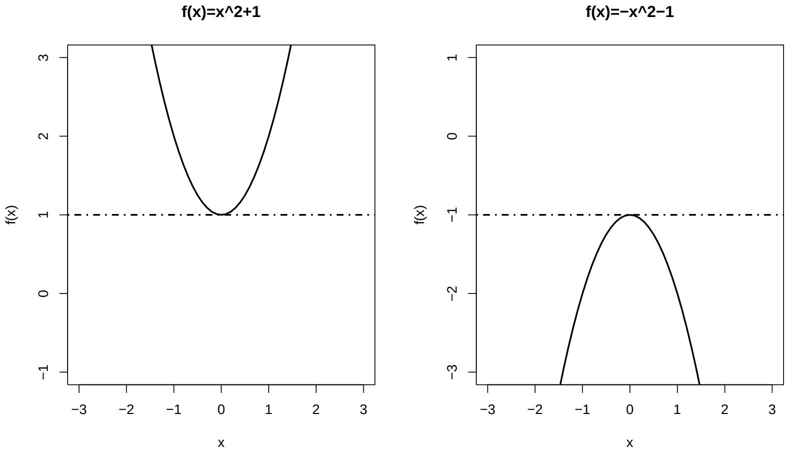
A **critical value** occurs when the behavior of a function changes.

Occurs where the derivative is equal to zero, i.e. goes from positive to negative or negative to positive.

Why do we care? We may want to find the maximum or minimum of a function (Maximum Likelihood Estimation).

- **maximum**: where a function stops increasing and starts to decrease.
- **minimum**: where a function stops decreasing and starts increasing.

Critical Values



Critical Values

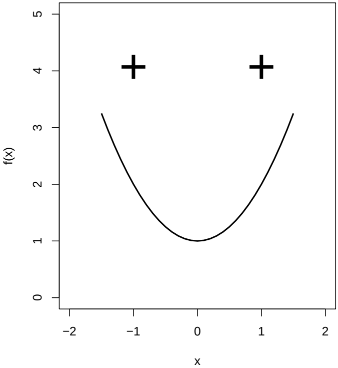
We can use the first derivative to find the critical point by setting it equal to zero and then solving for x, the root. The goal is to find x such that $f'(x) = 0$.

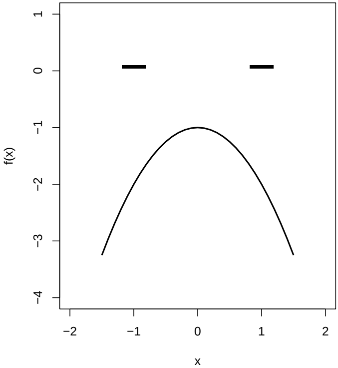
However, as seen on the previous slide, the derivative is zero for maximums **and** minimums. How do we tell the difference?

We use the second derivative.

Critical Values

For the max, the derivative decreases from positive to negative, so the second derivative will be negative. For the min, the derivative increases from negative to positive, so the second derivative will be positive.





Critical Values

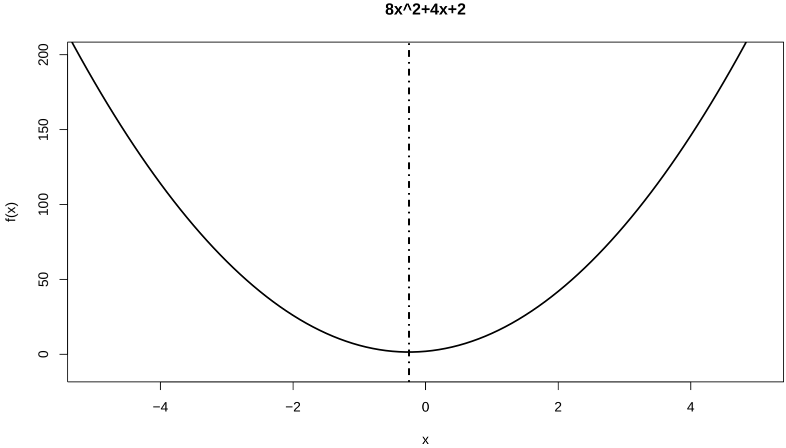
Examples

$$
\begin{aligned}
f(x) &= 8x^2 + 4x + 2 \\
f'(x) &= 16x + 4 \\
0 &= 16x + 4 \Rightarrow 16x = -4 \Rightarrow x = \frac{-1}{4} \\
f''(x) &= 16
\end{aligned}
$$

The critical value is at $x = -1/4$ and the second derivative is positive, so it is a minimum.

Critical Values

Examples



Critical Values

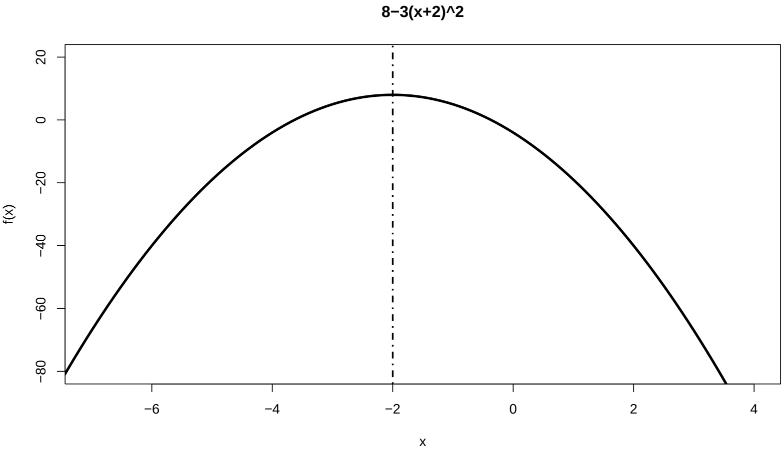
Examples

$$
\begin{aligned}
f(x) &= 8 - 3(x+2)^2 \\
f'(x) &= -6(x+2) \\
0 &= -6x - 12 \Rightarrow -6x = 12 \Rightarrow x = -2 \\
f''(x) &= -6
\end{aligned}
$$

The cricital value is at $x = -2$ and the second derivative is negative, so it is a maximum.

Critical Values

Examples



Critical Values

Saddle Points

If $f''(x) = 0$, then you have a **saddle point**. This is a critical point where the overall behavior of your function does not change.
For example: $f(x) = x^3$, $f'(x) = 3x^2$, $f''(x) = 6x$. At $x = 0$, we have $f'(x) = f''(x) = 0$.

Some functions have more than one maximum or minimum.

We call the largest maximum or the lowest minimum the **global** critical point. All others are referred to as **local** critical points. When looking, ideally we want to find the global maximum or minimum.

Critical Values

Global vs. Local