Center for Statistics and the Social Sciences Math Camp 2021

Lecture 4: Integral Calculus

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Outline

- Motivation for Integrals
- Rules of Integration
- Lots of Examples

Motivation for Integrals in Statistics

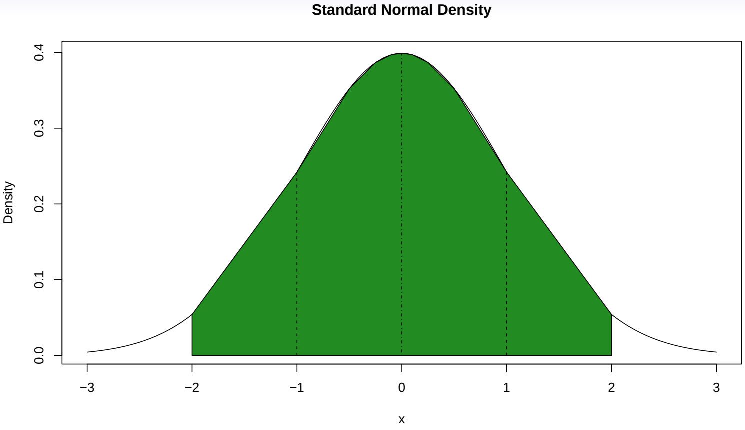


Figure: Standard Normal Density (N(0,1)). Approximately 68% of the probability lies within 1 standard deviation and 95% within 2 standard deviations. The area under the whole curve (from $-\infty$ to ∞) is 1.

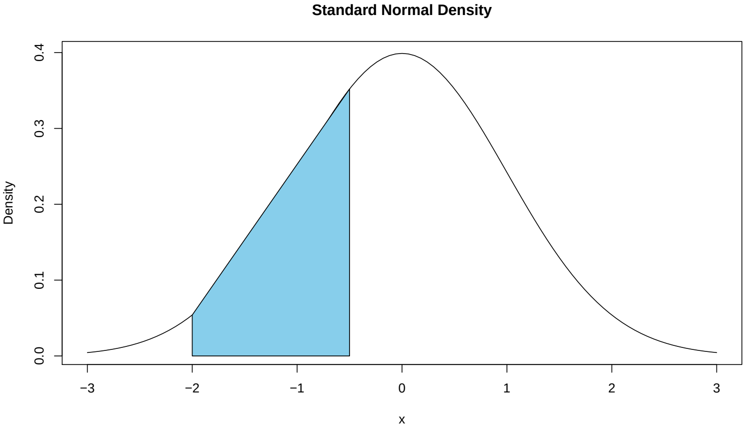
Motivation for Integrals in Statistics

Integral calculus...

- is a tool for computing areas under curves.
- can be used to compute percentile rankings.
- is also used to compute probabilities of events.
- will be needed to compute the expected values and variance of probability distributions.
- is heavily used in statistical theory!

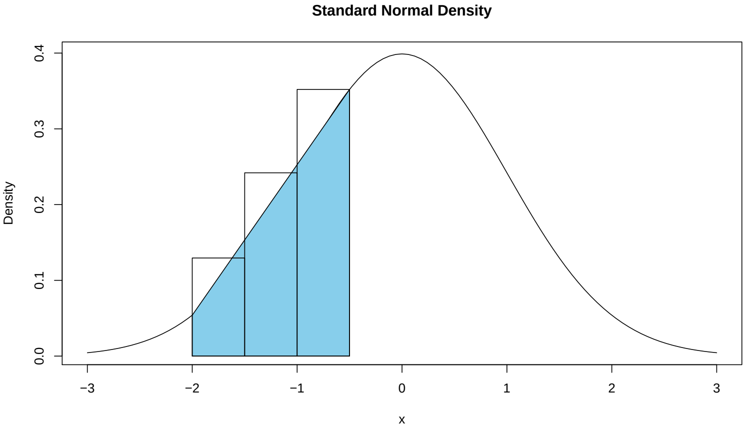
Motivation for Integrals in Statistics

What if we wanted to find the area under the curve from -2 to -0.5?



Motivation for Integrals in Statistics

We could approximate with rectangles or trapezoids. Narrower rectangles would give better approximations.



Differentiation Example

distance, velocity, acceleration

Let's take d=distance, v=velocity, a=acceleration. You may remember from physics, the distance travel after time t

$$d(t) = \frac{a}{2}t^2$$

The velocity at any time t is the instantaneous rate of change of the distance, $v(t) = d'(t)$:

$$v(t) = 2 \cdot \frac{a}{2}t = at$$

The acceleration at any time t is the instantaneous rate of change of the velocity, $a(t) = v'(t) = d''(t)$:

$$a(t) = a$$

Distance

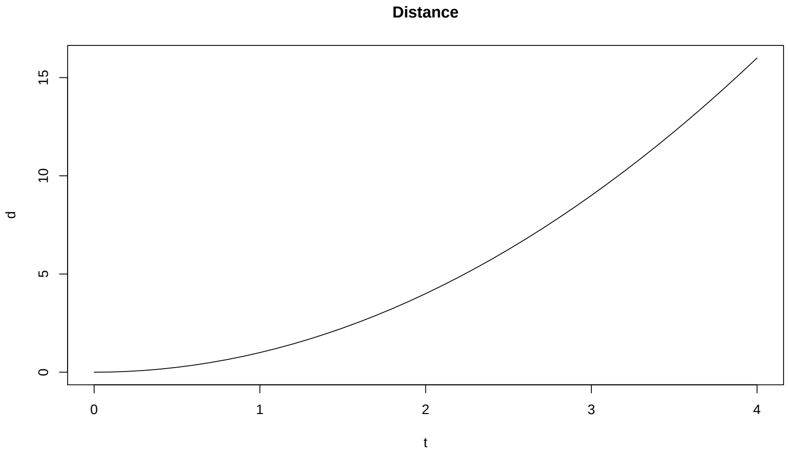


Figure: Distance over time, when $a(t) = 2$, $v(t) = 2t$, and $d(t) = t^2$.

Velocity

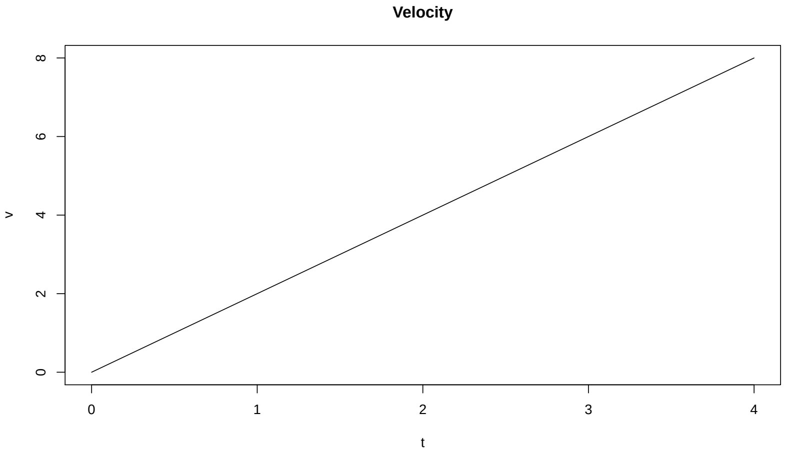


Figure: Velocity over time, when $a(t) = 2$, $v(t) = 2t$, and $d(t) = t^2$.

Acceleration

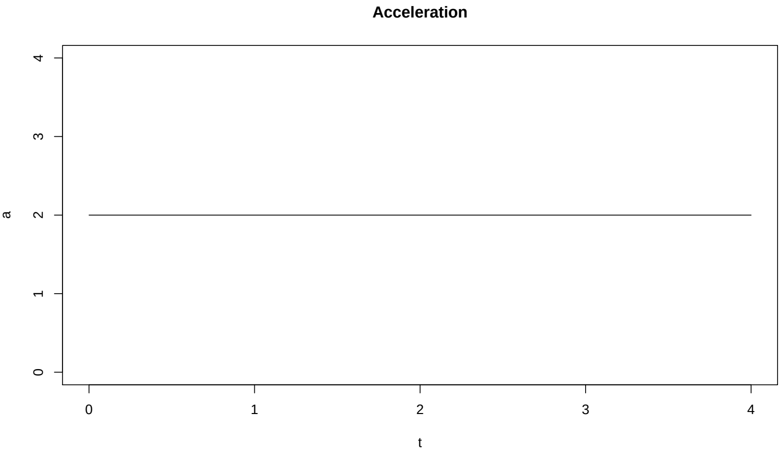


Figure: Acceleration over time, when $a(t) = 2$, $v(t) = 2t$, and $d(t) = t^2$.

What is the velocity at t=3 when a=2?

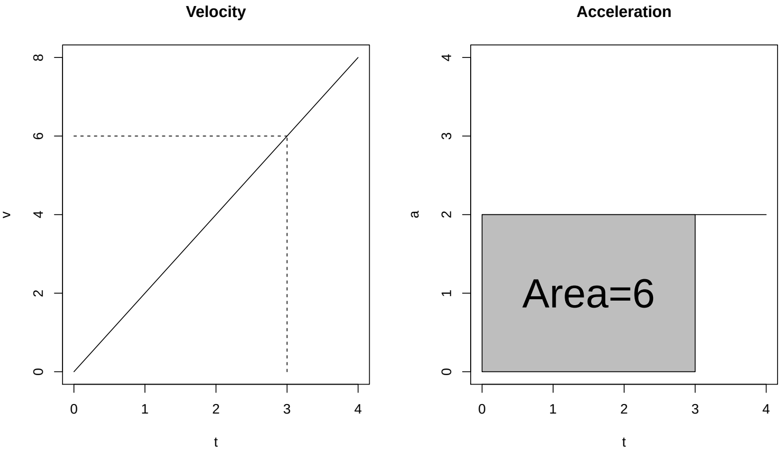
We know that $v(t) = 2t$, so clearly

$$v(3) = 2 \cdot 3 = 6.$$

However we can also find the velocity, by looking at the area under the acceleration curve from $t = 0$ to $t = 3$. This would just be the area of a rectangle (`base X height`),

$$(3 - 0) \cdot 2 = 3 \cdot 2 = 6.$$

What is the velocity at t=3 when a=2?



What is the distance at t=3 when a=2?

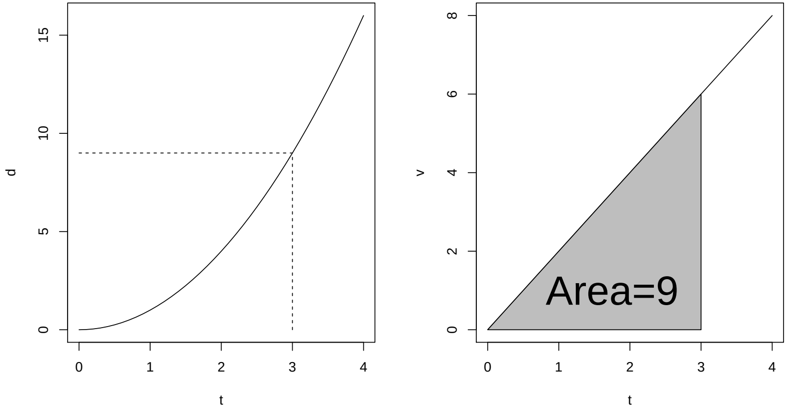
We know that $d(t) = 2/2t^2 = t^2$, so clearly

$$d(3) = 3^2 = 9.$$

However we can also find the distance, by looking at the area under the velocity curve from $t = 0$ to $t = 3$. This would just be the area of a triangle (1/2 `X base X height`),

$$1/2 \cdot (3 - 0) \cdot 6 = 3/2 \cdot 6 = 18/2 = 9.$$

What is the distance at t=3 when a=2?



Integration

The area under a curve is written:

$$\int_a^b f(x)dx$$

This formula is called the **definite integral** of $f(x)$ from a to b.

Here a and b are our endpoints of interest. You can think of the integral as the 'opposite' of the derivative.

More specifically,

$$\int_a^b f(x)dx = F(b) - F(a) \text{ where } F'(x) = f(x)$$

$F(x)$ is called the **indefinite integral** of $f(x)$. The important relationships between derivatives and integrals are:

$$F'(x) = f(x) \quad \& \quad \int f(x)dx = F(x)$$

What is an integral?

You can think of integrating as looking at a derivative and trying to find the original function.

- $\int 3dx$. What function has a derivative equal to 3? $3x$.
- $\int 2xdx$. What function has a derivative equal to $2x$? x^2.
- $\int e^x dx$. What function has a derivative equal to e^x? e^x.

In practice, you don't have to search for the right function. We have handy shortcuts (rules).

Integration Rules

Integrating a Constant

$$\int c dx = cx$$

Examples:

- $\int 1 dx = x$
- $\int 6 dx = 6x$
- $\int y dx = yx$

Integration Rules

Integrating a Power of x

$$\int x^n dx = \frac{1}{n+1} x^{n+1}$$

Examples:

- $\int x dx = \frac{1}{2} x^2$
- $\int \frac{1}{x^2} dx = \int x^{-2} dx = \frac{1}{-1} x^{-1} = -\frac{1}{x}$

Integration Rules

Integrating an Exponential and Logarithmic Functions

Exponential:

$$\int e^x dx = e^x$$

(Natural) Logarithm:

$$\int \frac{1}{x} dx = log(x)$$

Remember, $\frac{d}{dx}cos(x) = -sin(x)$, thus

$$\int sin(x)dx = -cos(x)$$

and $\frac{d}{dx}sin(x) = cos(x)$, thus

$$\int cos(x)dx = sin(x).$$

Integration Rules

Multiple of a Function

$$\int af(x)dx = a \cdot \int f(x)dx = aF(x)$$

Examples:

- $\int 4x^2 dx = 4\int x^2 dx = 4\left(\frac{1}{3}x^3\right) = \frac{4}{3}x^3$
- $\int \frac{3}{x^2} dx = 3\int \frac{1}{x^2} dx = 3\int x^{-2} dx = \frac{3}{-1}x^{-1} = -\frac{3}{x}$
- $\int \mu y dy = \mu \int y dy = \mu\left(\frac{1}{2}y^2\right) = \frac{\mu}{2}y^2$

$$\int (f(x) + g(x))\, dx = \int f(x)dx + \int g(x)dx = F(x) + G(x)$$

Examples:

- $\int 4x + 3x^2 dx = \int 4xdx + \int 3x^2 dx = 4\int xdx + 3\int x^2 dx = 4 \cdot \frac{1}{2}x^2 + 3 \cdot \frac{1}{3}x^3 = 2x^2 + x^3$
- $\int e^x - \frac{2}{x}dx = \int e^x dx - 2\int \frac{1}{x}dx = e^x - 2log(x)$

Sometimes the function we are integrating is similar to a simpler function with an easy derivative.

For example, $\int \frac{1}{1-x} dx$ is similar to $\int \frac{1}{x} dx$ which we know is $log(x)$. Similar to the chain rule, we can think about functions within functions.

Let's set $u = 1 - x$. If we differentiate the left with respect to u and the right with respect to x we have $du = -1dx$. Solving for dx we have $dx = -1du$. Now we can substitute these values into our original integral.

$$\int \frac{1}{1-x} dx = \int \frac{1}{u} \cdot (-1) du = -1 \int \frac{1}{u} du$$

Integration Rules

u-substitution continued

Now let's take the integral with respect to u:

$$\int \frac{1}{1-x} dx = -1 \int \frac{1}{u} du = -log(u)$$

Then we can plug in the value for $u = 1 - x$:

$$\int \frac{1}{1-x} dx = -1 \int \frac{1}{u} du = -log(u) = -log(1-x)$$

Example:

$$\int (2x+4)^3 dx$$

We can take $u = 2x + 4$. Then $du = 2dx$ or $\frac{1}{2}du = dx$.

When we make the substitutions in our integral we have:

$$\int (2x+4)^3 dx = \int u^3 \cdot \frac{1}{2} du = \frac{1}{2} \int u^3 du$$

Now we have an integral we can easily compute

$$\frac{1}{2} \int u^3 du = \frac{1}{2} \cdot \frac{1}{4} u^4 = \frac{1}{8} u^4$$

and then we just need to substitute back in for the functions of x.

$$\int (2x+4)^3 dx = \frac{1}{2} \int u^3 du = \frac{1}{8} u^4 = \frac{1}{8} (2x+4)^4$$

Finding Definite Integrals

Often we will be interested in knowing the exact area under the curve $f(x)$, not just the function $F(x)$.

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$$

Examples:

- $\int_0^1 x^2 dx = \frac{1}{3}x^3|_0^1 = \frac{1}{3}1^3 - \frac{1}{3}0^3 = \frac{1}{3}$
- $\int_0^\infty e^{-x} dx = -e^{-x}|_0^\infty = -e^{-\infty} - -e^0 = -\frac{1}{e^\infty} + e^0 = 1$
- $\int_2^8 \frac{1}{x} dx = log(x)|_2^8 = log(8) - log(2) = log\left(\frac{8}{2}\right) = log(4)$

Integration Example

distance, velocity, acceleration

Back to our original example, with $a = 2$. The velocity at any time $t = 3$ is the definite integral of of the acceleration, $v(3) = \int_0^3 a(t)dt$:

$$v(3) = \int_0^3 2dt = 2t|_0^3 = 2 \cdot 3 - 2 \cdot 0 = (3 - 0) \cdot 2 = 6$$

Similarly, the distance at any time $t = 3$ is the definite integral of of the velocity, $d(3) = \int_0^3 v(t)dt$:

$$d(3) = \int_0^3 v(t)dt = \int_0^3 2tdt = t^2|_0^3 = 3^2 - 0^2 = 9$$

Example

$$\int_0^3 e^{x/3} dx$$

We could take $u = x/3$. Then $du = 1/3dx$ and $3du = dx$.

When we substitute in for u and dx it is important to note that we must also substitute in for our limits of integration. The lower value $u = 0/3 = 0$ and the upper value would be $u = 3/3 = 1$.

$$\int_0^3 e^{x/3} dx = \int_0^1 e^u \cdot 3du = 3 \int_0^1 e^u du = 3e^u|_0^1 = 3(e^1 - e^0) = 3(e - 1)$$

The End

Questions?